MATHEMATICS

ON MARKOV CHAINS AND INTUITIONISM. IV

BY

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0. In three foregoing papers [1], [2] and [3] (Numbers in brackets refer to the list of references at the end of this paper) Markov Chains with stationary transition probabilities were discussed from the intuitionistic point of view. In those papers the state space was enumerable and the time-parameter was discrete resp. continuous.

The purpose of this paper is to give an intuitionistic treatment of Markov Chains with a finite state space.

For the intuitionistic nomenclature and terminology the reader is referred to [4].

1. *Basic assumptions*

Let Ω be a species consisting of $N \geqq 1$ mathematical entities. The elements of Ω will be indicated by $E_1, E_2, \ldots, E_N$, which entities are called the states of the Markov Chain.

For every state $E_i \in \Omega (1 \leqq i \leqq N)$ a real number p_i (the absolute probability at 0) is given such that

α_1: $$0 \not> p_i \not> 1,$$

α_2: $$\sum_{i=1}^{N} p_i = 1.$$

Furthermore it is supposed that for every ordered pair E_i, E_j ($i=j$ included) a number p_{ij} is given such that:

β_1: $$0 \not> p_{ij} \not> 1,$$

β_2: $$\sum_{j=1}^{N} p_{ij} = 1 \qquad (i = 1, 2, \ldots, N).$$

The numbers p_{ij} are called the one-step transition probabilities and the n-step transition probabilities $p_{ij}^{(n)}$ are defined in the usual way by

$$p_{ij}^{(1)} = p_{ij}$$

$$p_{ij}^{(n+1)} = \sum_{k=1}^{N} p_{ik} p_{kj}^{(n)} \qquad (n = 1, 2, \ldots).$$

The absolute probability $p_i^{(n)}$ of state E_i at time n is then given by

$$p_i^{(n)} = \sum_{k=1}^{N} p_k p_{ki}^{(n)} \qquad (n = 1, 2, \ldots).$$

2.1 Definitions.

The state E_j is

an α-consequent of E_i if $\quad (\exists n)(p_{ij}^{(n)} > 0)$

a β-consequent of E_i if $\quad (\exists n)(p_{ij}^{(n)} \neq 0)$

a γ-consequent of E_i if $\neg\neg (\exists n)(p_{ij}^{(n)} > 0)$.

These relations between the states E_i and E_j will be indicated by $E_i \xrightarrow{\alpha} E_j$, $E_i \xrightarrow{\beta} E_j$ and $E_i \xrightarrow{\gamma} E_j$ respectively.

If $E_i \xrightarrow{\alpha} E_j$ such that $p_{ij}^{(n)} > 0$ then E_j is called an α-consequent of E_i of order n and E_j is a β-consequent of order n if $p_{ij}^{(n)} \neq 0$. The notion of order is not defined in the case $E_i \xrightarrow{\gamma} E_j$.

Let us abbreviate "$p_{ij}^{(n)} > 0$" by $A(i, j, n)$ then, as Prof. Dr. B. van Rootselaar wrote me, we have:

$$p_{ij}^{(n)} \neq 0 \Leftrightarrow \neg\neg A(i, j, n)$$

on account of $0 \not> p_{ij}$, and the consequent-predicates can be given in the form:

$$E_i \xrightarrow{\alpha} E_j \equiv (\exists n)\, A(i, j, n)$$
$$E_i \xrightarrow{\beta} E_j \equiv (\exists n) \neg\neg A(i, j, n)$$
$$E_i \xrightarrow{\gamma} E_j \equiv \neg\neg (\exists n)\, A(i, j, n) \equiv \neg\neg (E_i \xrightarrow{\alpha} E_j).$$

Remarks.

1. The notions of consequent as given above are equivalent from the classical point of view.

2. From the rule:

$$\neg\neg (\exists x)\, A(x) \Leftrightarrow \neg\neg (\exists x) \neg\neg A(x) \qquad \text{(cf. [5])}$$

we see that we do not get a new definition if we replace

$$\neg\neg (\exists n)(p_{ij}^{(n)} > 0) \text{ by } \neg\neg (\exists n)(p_{ij}^{(n)} \neq 0).$$

3. Note that if we want to prove that E_j is an α-consequent or a γ-consequent of E_i then it is no restriction to suppose that the natural number n as required in the definition satisfies the inequality $1 \leqq n \leqq N$.

This we see as follows (cf. [6], p. 176).

Let $n > N$ be a natural number such that $p_{ij}^{(n)} > 0$, then from

$$p_{ij}^{(n)} = \sum_{j_1, \ldots, j_{n-1}} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j}$$

it follows that a sequence $k_1, k_2, \ldots, k_{n-1}$ of natural numbers can be indicated such that

$$p_{ik_1} p_{k_1 k_2} \cdots p_{k_{n-1} j} > 0.$$

This sequence $k_1, k_2, \ldots, k_{n-1}$ contains equal numbers for $n>N$ and $1 \leqq k_r \leqq N$ $(r=1, 2, \ldots, N-1)$. Hence we can construct a sequence $m_1, m_2, \ldots, m_s$ by deleting $k_{\nu+1}, k_{\nu+2}, \ldots, k_{\nu+\mu}$ if $k_\nu = k_{\nu+\mu}$ such that

$$p_{im_1}\, p_{m_1 m_2} \ldots p_{m_s j} > 0 \text{ with } s \leqq N.$$

Applying the rule

$$[A \Rightarrow B] \Rightarrow [\neg\neg A \Rightarrow \neg\neg B] \qquad \text{(cf. [5])}$$

we see that if E_j is a γ-consequent of E_i then $1 \leqq n \leqq N$ is no restriction.

However, this method cannot be applied in the case that we only know $p_{ij}^{(n)} \neq 0$, for if x_1 and x_2 are two real numbers then from $x_1 + x_2 \neq 0$ it does not follow that the disjunction

$$(x_1 \neq 0) \vee (x_2 \neq 0)$$

can be proven.

2.2.1. The relations

$$E_i \xrightarrow{\alpha} E_j \Rightarrow E_i \xrightarrow{\beta} E_j$$

and

$$E_i \xrightarrow{\beta} E_j \Rightarrow E_i \xrightarrow{\gamma} E_j$$

are trivial (cf. [5]).

2.2.2. We have introduced three notions of "consequent". It is the purpose of this section to show by counterexamples that these notions are different from our point of view.

In the first example we define a transition matrix such that it can happen that E_j is a β-consequent of E_i without having a proof of E_j is an α-consequent of E_i, hence the notion of "β-consequent" is essentially weaker than "α-consequent".

In the second example we show that the notion of "E_j is a γ-consequent of E_i" is essentially weaker than "E_j is a β-consequent of E_i".

Example 1.

The following method of constructing a real number has been given by Brouwer ([7] cf. [8]. Both papers are discussed in [4], chapter VIII).

Let τ be a mathematical proposition which has not yet been tested, i.e. neither $\neg\tau$ nor $\neg\neg\tau$ has been proved.

A real number ε is defined as follows:

We construct a sequence $\{\varepsilon_n\}$ of rational numbers by choosing

$$\varepsilon_n = 2^{-n}$$

as long as τ has not been tested, but if τ is tested between the choice of $\varepsilon_m = 2^{-m}$ and ε_{m+1}, then we choose:

$$\varepsilon_{m+n} = 2^{-m} \text{ for every } n.$$

By

$$\varepsilon = \lim_{n\to\infty} \varepsilon_n$$

a real number is defined.

It is easily seen that $\varepsilon \neq 0$, but we have no proof of $\varepsilon \# 0$.

Let Ω_1 be a state space consisting of two states E_1 and E_2 and consider a Markov Chain with

$$P_1 = \begin{pmatrix} \varepsilon & 1-\varepsilon \\ 0 & 1 \end{pmatrix}$$

as its one-step transition matrix. A simple calculation gives

$$P_1^{(n)} = \begin{pmatrix} \varepsilon^n & 1-\varepsilon^n \\ 0 & 1 \end{pmatrix},$$

hence $E_1 \xrightarrow{\beta} E_1$ and E_1 is a β-consequent of E_1 of order 1, but we have no proof of $E_1 \xrightarrow{\alpha} E_1$, for this requires a proof of $\varepsilon \# 0$, which is not known.

Example 2.

Again a mathematical proposition τ is considered which has not been tested. We define the real numbers a, b and c by

$$a = \lim_{n\to\infty} a_n \quad ; \quad b = \lim_{n\to\infty} b_n \quad \text{and} \quad c = \lim_{n\to\infty} c_n,$$

where the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are constructed as follows:

As long as τ has not been tested we choose:

$$a_n = b_n = c_n = 2^{-n},$$

but if τ is tested between the choices of $a_m = b_m = c_m$ and a_{m+1}, b_{m+1} and c_{m+1}, then we choose:

$$\begin{array}{ll} a_{m+n} = 2^{-m},\ b_{m+n} = c_{m+n} = 0 & \text{for every } n \text{ if } m = 3k, \\ a_{m+n} = c_{m+n} = 0,\ b_{m+n} = 2^{-m} & \text{for every } n \text{ if } m = 3k+1, \\ a_{m+n} = b_{m+n} = 0,\ c_{m+n} = 2^{-m} & \text{for every } n \text{ if } m = 3k+2, \end{array}$$

where k is a natural number.

Note that $a+b+c \neq 0$, but $ab = bc = ac = 0$.

Let Ω_2 be a state space consisting of the states E_1, E_2, E_3 and E_4 and let us consider a Markov Chain with

$$P_2 = \begin{pmatrix} a & b & c & 1-a-b-c \\ a & a & a & 1-3a \\ 0 & c & 0 & 1-c \\ a & a & a & 1-3a \end{pmatrix}$$

as its one-step transition matrix.

A straightforward calculation gives:

$$p_{12} = b \quad ; \quad p_{12}^{(2)} = c^2 + a(1-a) \quad ; \quad p_{12}^{(3)} = a(1-a)^2$$

$$p_{12}^{(4)} = a(1-a)(1-a+2a^2).$$

Evidently:

$$\neg\neg\, (\exists n)(p_{12}^{(n)} > 0),$$

hence $E_1 \xrightarrow{\gamma} E_2$, but as long as τ has not been tested we have no proof of $E_1 \xrightarrow{\beta} E_2$. This we see as follows:

Let us suppose that we have a natural number n such that we have a proof of

$$p_{12}^{(n+1)} \neq 0 \tag{1}$$

From the foregoing calculations it is clear that $n \geqq 4$. The relation (1) tells us:

$$\sum_{j_i=1,2,3,4} p_{1j_1} p_{j_1 j_2} \dots p_{j_n 2} \neq 0,$$

where the possible values of j_n are 1, 2, 3 and 4.

If $j_n = 1$ then with the term $p_{1j_1} p_{j_1 j_2} \dots p_{12}$ corresponds the sequence

$$E_1, E_{j_1}, E_{j_2}, \dots, E_i, E_1, E_2$$

of states, but a simple inspectation of the matrix P_2 tells us:

$$p_{i1} p_{12} = p_{i1} b = 0 \qquad (\text{for } ab = 0).$$

If $j_n = 2$, then we have as the corresponding sequence:

$$E_1, E_{j_1}, \dots, E_i, E_2, E_2$$

with $p_{22} = a$ and this gives that

$$p_{1j_1} p_{j_1 j_2} \dots p_{i2} p_{22} = af(a, b, c),$$

where $f(a, b, c)$ is a polynomium, which implies that we can put

$$af(a, b, c) = g_1(a) \qquad (\text{for } ab = ac = 0),$$

such that each term of $g_1(a)$ contains a as a factor.

In the same way we see that $j_n = 3$ gives a polynomium $g_2(c)$ and $j_n = 4$ leads to a polynomium $g_3(a)$.

Combining these results we see:

$$p_{12}^{(n)} = \varphi_n(a, c),$$

where $\varphi_n(a, c)$ has only terms with either a or c as a factor. This implies that as long as the proposition τ has not been tested we have no proof of $p_{12}^{(n)} \neq 0$ for every $n = 1, 2, \dots$, i.e. we have no proof of $(\exists n)(p_{12}^{(n)} \neq 0)$, hence we can only say that E_2 is a γ-consequent of E_1, but a proof of "E_2 is a β-consequent of E_1" is not known.

2.3.1. Definitions.

The states E_i and E_j satisfy the relation

$$\begin{array}{ll} c_\alpha \text{ if} & (\exists n, m)\,[(p_{ij}^{(n)} > 0) \wedge (p_{ji}^{(m)} > 0)]; \\ c_\beta \text{ if} & (\exists n, m)\,[(p_{ij}^{(n)} \neq 0) \wedge (p_{ji}^{(m)} \neq 0)]; \\ c_\gamma \text{ if} & \neg\neg\,(\exists n, m)\,[(p_{ij}^{(n)} \wedge 0) \wedge (p_{ji}^{(m)} > 0)]. \end{array}$$

These relations will be indicated by $E_i\, c_\alpha\, E_j$, $E_i\, c_\beta\, E_j$ and $E_i\, c_\gamma\, E_j$ respectively and the states E_i and E_j are called α-, β-, resp. γ-communicating.

It is easily seen that these relations are transitive and symmetric, but they are not reflexive in general. As to c remark 2 of section 2.1. can be repeated.

2.3.2. Of course we have:

$$E_i\, c_\alpha\, E_j \quad \Rightarrow \quad E_i\, c_\beta\, E_j,$$

$$E_i\, c_\beta\, E_j \quad \Rightarrow \quad E_i\, c_\gamma\, E_j \quad \text{and}$$

$$E_i\, c_\gamma\, E_j = \neg\neg\, (E_i\, c_\alpha\, E_j).$$

2.3.3. Definitions.

We shall say that the states E_i and E_j satisfy the relations

$$\begin{array}{ll}
c_\alpha & \text{if } (\exists n, m)\, [A(i,j,n) \wedge A(j,i,m)]; \\
c(1) & \text{if } (\exists n)\, \neg\neg\, A(i,j,n) \wedge (\exists m)\, A(j,i,m); \\
c(2) & \text{if } (\exists n)\, A(i,j,n) \wedge (\exists m)\, \neg\neg\, A(j,i,m); \\
c_\beta & \text{if } (\exists n, m)\, \neg\neg\, [A(i,j,n) \wedge A(j,i,m)]; \\
c(3) & \text{if } \neg\neg\, (\exists n)\, A(i,j,n) \wedge (\exists m)\, A(j,i,m); \\
c(4) & \text{if } (\exists n)\, A(i,j,n) \wedge \neg\neg\, (\exists m)\, A(j,i,m); \\
c(5) & \text{if } \neg\neg\, (\exists n)\, A(i,j,n) \wedge (\exists m)\, \neg\neg\, A(j,i,m); \\
c(6) & \text{if } (\exists n)\, \neg\neg\, A(i,j,n) \wedge \neg\neg\, (\exists m)\, A(j,i,m); \\
c_\gamma & \text{if } \neg\neg\, (\exists n, m)\, [A(i,j,n) \wedge A(j,i,m)],
\end{array}$$

where "$p_{ij}^{(n)} > 0$" is abbreviated by $A(i,j,n)$.

These relations will be used in the form $E_i\, c_\alpha\, E_j$, $E_i\, c(k)\, E_j$ etc. and if $E_i\, c_\alpha\, E_j$ then we shall say that E_i and E_j are α-communicating and in an analogous way in the other cases.

The relations c_α, c_β and c_γ are transitive and symmetric, but they are not reflexive in general, but observe that the $c(k)$-relations ($k = 1, 2, 4, 5, 6$) are not symmetric.

Evidently we have:

$$E_i\, c(1)\, E_j \Leftrightarrow E_j\, c(2)\, E_i;$$

$$E_i\, c(3)\, E_j \Leftrightarrow E_j\, c(4)\, E_i;$$

$$E_i\, c(5)\, E_j \Leftrightarrow E_j\, c(5)\, E_i;$$

$$[E_i\, c(k)\, E_j] \wedge [E_j\, c(k)\, E_i] \Leftrightarrow E_i\, c_\alpha\, E_j \quad \text{for } k = 1, 2, 3, 4;$$

$$[E_i\, c(5)\, E_j] \wedge [E_j\, c(5)\, E_i] \Leftrightarrow E_i\, c_\beta\, E_j.$$

The notions given above are interrelated which is expressed by the following scheme:

$$\begin{tikzcd}
 & c(1) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & c(3) \arrow[r, Rightarrow] & c(5) \arrow[dr, Rightarrow] & \\
c_\alpha \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & c(\beta) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & c_\gamma \\
 & c(2) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & c(4) \arrow[r, Rightarrow] & c(6) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

In the following sections we shall restrict ourselves to the symmetric relations.

2.3.4. Definitions.

The state E_i is called

$$\begin{aligned}
&\alpha\text{-essential if} \quad (j)[\{E_i \xrightarrow{\alpha} E_j\} \Rightarrow \{E_j \xrightarrow{\alpha} E_i\}],\\
&\beta\text{-essential if} \quad (j)[\{E_i \xrightarrow{\beta} E_j\} \Rightarrow \{E_j \xrightarrow{\beta} E_i\}],\\
&\gamma\text{-essential if} \quad (j)[\{E_i \xrightarrow{\gamma} E_j\} \Rightarrow \{E_j \xrightarrow{\gamma} E_i\}],\\
&\text{inessential if} \quad (\exists j)[\{E_i \xrightarrow{\gamma} E_j\} \wedge (n)(p_{ji}^{(n)}=0)].
\end{aligned}$$

2.3.5. In classical mathematics a state E_i is called essential (cf. [9], p. 11) if E_i is a consequent of each of its consequents. This definition makes no troubles from the classical point of view for then one can always decide whether a state E_j is a consequent of E_i or not.

However, from the intuitionistic point of view the situation is more complicated for we have more consequent notions which are not equivalent.

2.3.6. If the state E_i is an α-essential state, then it is a γ-essential state. If the state E_i is a γ-essential state then it is impossible, that it is not an α-essential state.

Proof. On account of the rules given in [5] and in [4], chapter VII we have the implications:

$$\begin{array}{llll}
 & (j) & [(\exists n)(p_{ij}^{(n)}>0) \Rightarrow & (\exists m)(p_{ji}^{(m)}>0)] \\
(1) & \Rightarrow \neg\neg\, (j) & [(\exists n)(p_{ij}^{(n)}>0) \Rightarrow & (\exists m)(p_{ji}^{(m)}>0)] \\
(2) & \Rightarrow \quad (j)\, \neg\neg & [(\exists n)(p_{ij}^{(n)}>0) \Rightarrow & (\exists m)(p_{ji}^{(m)}>0)] \\
(3) & \Rightarrow \quad (j)\, [\neg\neg & (\exists n)(p_{ij}^{(n)}>0) \Rightarrow \neg\neg & (\exists m)(p_{ji}^{(m)}>0)],
\end{array}$$

which proves the first statement.

The proof of the second statement is obtained from the implications:

$$(3) \Rightarrow (2) \Rightarrow (1).$$

The implication $(2) \Rightarrow (1)$ is not trivial for it is not provable in general, but in our case the variable j can take on only the finite number of values $1, 2, \ldots, N$.

Remark.

Prof. Dr. B. van Rootselaar observed that an α-essential state need not to be a β-essential state.

2.3.7. Counterexample.

The counterexample we shall discuss now is intended to show that the notions of being α-, β-, γ-essential are not equivalent and to illustrate that the three kinds of states can appear together in a Markov Chain.

To construct such an example we consider two real numbers p and q such that

$$[p \nless 0] \wedge [q \nless 0] \wedge [p+q \neq 0] \wedge [pq=0],$$

however, we suppose that we have no proof of

$$(p>0) \vee (q>0).$$

Such numbers can be constructed by the method as given in section 2.2.2., example 2.

We consider a state space Ω consisting of the states E_1, E_2 and E_3 and let

$$P_3 = \begin{pmatrix} p & q & 1-p-q \\ p+q & 0 & 1-p-q \\ p+q & 0 & 1-p-q \end{pmatrix}$$

be the one-step transition matrix of a Markov Chain, hence with

$$P_3^{(2)} = \begin{pmatrix} p+q & 0 & 1-p-q \\ p+q-q^2 & q^2 & 1-p-q \\ p+q-q^2 & q^2 & 1-p-q \end{pmatrix} \quad \text{and}$$

$$P_3^{(3)} = \begin{pmatrix} p+q-q^2 & q^2 & 1-p-q \\ p+q-q^2+q^3 & q^2-q^3 & 1-p-q \\ p+q-q^2+q^3 & q^2-q^3 & 1-p-q \end{pmatrix}$$

as its two-, resp. three-step transition matrix.

From these matrices we see:

E_1 is β-essential, but we have no proof of: E_1 is α-essential, for we have no proof of

$$[E_1 \xrightarrow{\alpha} E_3] \Rightarrow [E_3 \xrightarrow{\alpha} E_1].$$

E_2 is neither α-, nor β-, nor γ-essential.

E_3 is α-essential though we have neither a proof of $E_3 \xrightarrow{\alpha} E_1$, nor a proof of $E_3 \xrightarrow{\alpha} E_2$. The state E_1 is a β-consequent of E_3 and the same is true for the state E_2.

2.3.8. Let $C_\alpha(i)$ be defined by:

$$C_\alpha(i) = \{E_j \mid E_j \, c_\alpha \, E_i\}.$$

In the same way the species $C_\beta(i)$ and $C_\gamma(i)$ are introduced.

From section 2.3.2. it becomes clear that:

$$C_\alpha(i) \subset C_\beta(i) \subset C_\gamma(i).$$

Theorem: $\neg\neg\, [C_\alpha(i) = C_\gamma(i)]$.

Proof. We have to prove:

$$\neg\neg\, (j)(E_i\, c_\alpha\, E_j \Leftrightarrow E_i\, c_\gamma\, E_j).$$

Evidently we have $(j)(E_i\, c_\alpha\, E_j \Rightarrow E_i\, c_\gamma\, E_j)$.

For the proof of the second implication we suppose that

$$E_j \in C_\gamma(i) \wedge E_j \notin C_\alpha(i),$$

then we have:

$$[E_j\, c_\gamma\, E_i] \wedge [E_j\, c_\alpha\, E_i] \text{ i.e.}$$

$$\neg\neg\, (\exists n, m)\, [(p_{ij}^{(n)} > 0) \wedge (p_{ji}^{(m)} > 0)] \wedge \neg\, (\exists n, m)\, [(p_{ij}^{(n)} > 0) \wedge (p_{ji}^{(m)} > 0)]$$

hence we have obtained a contradiction. This implies:

$$\begin{aligned} (j) \quad & (E_i\, c_\gamma\, E_j \Rightarrow \neg\neg\, E_j\, c_\alpha\, E_i) \Rightarrow \\ (j)\, \neg\neg\, & (E_i\, c_\gamma\, E_j \Rightarrow E_j\, c_\alpha\, E_i) \Rightarrow \\ \neg\neg\, (j) \quad & (E_i\, c_\gamma\, E_j \Rightarrow E_j\, c_\alpha\, E_i), \end{aligned}$$

These implications are motivated by the rules given in [5] and by the fact that j runs through a finite spread.

2.3.9. The notion $C_\alpha(i)$, $C_\beta(i)$ and $C_\gamma(i)$ are equivalent from the classical point of view and are indicated by $C(i)$ (cf. [9], p. 11).

These classes $C(i)$ are fundamental because in classical mathematics one can prove: Two classes $C(i)$ and $C(j)$ coincide or are disjoint. In [1] a counterexample is given from which we see that we cannot prove this theorem from our point of view. In the same paper it is shown that the condition

$$(p_{ij} = 0) \vee (p_{ij} \# 0) \text{ for every pair } E_i, E_j$$

is necessary and sufficient for a finite Markov Chain in order to prove: the species $C_\alpha(i)$ and $C_\alpha(j)$ coincide or are disjoint.

It is easily seen that the α-, β- and γ-notions coincide if we suppose:

$$(p_{ij} = 0) \vee (p_{ij} \# 0) \text{ for every pair } E_i, E_j.$$

This statement is not true if the finitenes of the chain has not been given.

3. *Some limitproperties*

3.1 In this section we need the lemma:

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that

$$(n)(a_n \nless a_{n+1} \nless b_{n+1} \nless b_n)$$

and

$$\lim_{n \to \infty} (a_n - b_n) = 0,$$

then we have:

$$\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} b_n \quad \text{exist and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

The proof is simple and will be omitted.

3.2. Doob ([6], p. 173–174) proves the theorem:

If there exists an integer $t \geqq 1$ and a set $J = \{j_1, \ldots, j_{N_1}\}$ of $N_1 \geqq 1$ values of j such that

$$\min_{1 \leqq i \leqq N} (p_{ij_1}^{(t)}, p_{ij_2}^{(t)}, \ldots, p_{ij_{N_1}}^{(t)}) = \delta \neq 0$$

then there exist numbers $p_1, p_2, \ldots, p_N$ such that

$$\lim_{n\to\infty} p_{ij}^{(n)} = p_j \qquad (i, j = 1, 2, \ldots, N)$$

$$p_{j_i} \nless \delta \qquad (i = 1, 2, \ldots, N_1),$$

$$0 \ngtr p_j \ngtr 1 \quad (j = 1, 2, \ldots, N) \quad \text{and} \quad \sum_{j=1}^{N} p_j = 1.$$

Moreover:

$$|p_{ij}^{(n)} - p_j| \ngtr (1 - N\delta)^{(n/t)-1} \qquad (n > t).$$

With only slight refinements the proof as given by Doob can be rewritten and we shall do so to point out the intuitionistic difficulties. As to the condition $\delta \neq 0$ we suppose: $\delta > 0$.

Proof. Let the numbers $m_j^{(r)}$ and $M_j^{(r)}$ be defined by

$$m_j^{(r)} = \min_{1 \leqq i \leqq N} p_{ij}^{(r)} \quad ; \quad M_j^{(r)} = \max_{i \leqq i \leqq N} p_{ij}^{(r)}.$$

For fixed j we have the relations:

$$m_j^{(r+1)} = \min_{1 \leqq i \leqq N} \sum_{k=1}^{N} p_{ik} p_{kj}^{(r)} \nless \min_{1 \leqq i \leqq N} \sum_{k=1}^{N} p_{ik} m_j^{(r)} = m_j^{(r)},$$

hence the sequence $m_j^{(r)}$ is monotone and non-descending.

In the same way we get:

$$M_j^{(r+1)} \ngtr M_j^{(r)},$$

and these results lead to

$$m_j^{(1)} \ngtr m_j^{(2)} \ngtr \ldots \ngtr M_j^{(2)} \ngtr M_j^{(1)}.$$

Now we consider the relation

$$p_{ij}^{(n)} - p_{kj}^{(n)} = \sum_{s=1}^{N} [p_{is}^{(t)} - p_{ks}^{(t)}] p_{sj}^{(n-t)} \text{ with } n > t. \tag{1}$$

The differences $p_{is}^{(t)} - p_{ks}^{(t)}$, which occur in the sum of (1), are real numbers i.e. we can suppose that they are given by the sequences $\{\tau_{\alpha, iks}\}$ of intervals with rational endpoints and such that

$$\tau_{\alpha, iks} \supset \tau_{\alpha+1, iks}.$$

The length of the interval $\tau_{\alpha, iks}$ will be indicated by $d_{\alpha, iks}$ ($\alpha = 1, 2, \ldots$; $i, k, s = 1, 2, \ldots, N$) and it is allowed to suppose

$$(2) \qquad d_{\alpha, iks} < N^{-1} \cdot 2^{-\alpha}.$$

The numbers ${}_\alpha\beta_{iks}$, ${}_\alpha\beta'_{iks}$ and ${}_\alpha\gamma_{iks}$ are defined by

$$(3) \qquad {}_\alpha\beta_{iks} = p_{is}^{(t)} - p_{ks}^{(t)} \text{ if } 0 \notin \tau_{\alpha, iks} \text{ and if } p_{is}^{(t)} - p_{ks}^{(t)} > 0$$

$$(4) \qquad {}_\alpha\beta'_{iks} = p_{ks}^{(t)} - p_{is}^{(t)} \text{ if } 0 \notin \tau_{\alpha, iks} \text{ and if } p_{is}^{(t)} - p_{ks}^{(t)} < 0$$

$$(5) \qquad {}_\alpha\gamma_{iks} = p_{is}^{(t)} - p_{ks}^{(t)} \text{ if } 0 \in \tau_{\alpha, iks}.$$

Evidently:

$$(6) \qquad \sum_{s=1}^{N} [p_{is}^{(t)} - p_{ks}^{(t)}] = \sum_{(s_3)} {}_\alpha\beta_{iks} - \sum_{(s_4)} {}_\alpha\beta'_{iks} + \sum_{(s_5)} {}_\alpha\gamma_{iks} = 0,$$

where (s_3) resp. (s_4) and (s_5) indicates that we sum over those indices s for which (3) resp. (4) and (5) holds.

The relations (2) and (6) imply:

$$(7) \qquad \Big| \sum_{(s_3)} {}_\alpha\beta_{iks} - \sum_{(s_4)} {}_\alpha\beta'_{iks} \Big| < 2^{-\alpha}.$$

For every fixed α, i and k we can decide whether there exists an index s or not, which satisfies the condition mentioned in relation (3). Independent of the existence of such an index s we have:

$$(8) \qquad \sum_{(s_3)} {}_\alpha\beta_{iks} = \sum_{(s_3)} p_{is}^{(t)} - \sum_{(s_3)} p_{ks}^{(t)} \not> 1 - \sum_{(s_3)} p_{ks}^{(t)} \not> 1 - N_1\delta$$

on account of the assumptions of the theorem.

From
$$\begin{aligned} M_j^{(n+t)} - m_j^{(n+t)} &= \max_i p_{ij}^{(n+t)} - \min_i p_{ij}^{(n+t)} \\ &= \max_{i,k} \sum_{s=1}^{N} (p_{is}^{(t)} - p_{ks}^{(t)})\, p_{sj}^{(n)} \\ &< \max_{i,k} \Big[\sum_{(s_3)} {}_\alpha\beta_{iks} M_j^{(n)} - \sum_{(s_4)} {}_\alpha\beta'_{iks} m_j^{(n)} \Big] + 2^{-\alpha} \end{aligned}$$

it follows on account of (7) that for every $\alpha \geqq 1$ we have:

$$M_j^{(n+t)} - m_j^{(n+t)} < \max_{i,k} \sum_{(s_3)} {}_\alpha\beta_{iks} (M_j^{(n)} - m_j^{(n)}) + 2^{-\alpha+1},$$

hence (8) implies:

$$(9) \qquad M_j^{(n+t)} - m_j^{(n+t)} < (1 - N_1\delta)(M_j^{(n)} - m_j^{(n)}) + 2^{\alpha-+1}.$$

Relation (9) is true for every natural number α, hence

$$M_j^{(n+t)} - m_j^{(n+t)} \not> (1 - N_1\delta)(M_j^{(n)} - m_j^{(n)})$$

and this combined with

$$M_j^{(t)} - m_j^{(t)} \not> 1 - N_1\delta$$

gives

$$(10) \qquad M_j^{(kt)} - m_j^{(kt)} \not> (1 - N_1\delta)^{kt}.$$

Lemma 3.1. and relation (10) imply

$$\lim_{n\to\infty} M_j^{(n)} = \lim_{n\to\infty} m_j^{(n)} \overset{\mathrm{df}}{=} p_j$$

and the proof now runs without difficulties.

3.3.1. The following theorem can be proven from the classical point of view (cf. [6], p. 175).

If $P=(p_{ij})$ is the transition matrix of the Markov Chain then there exists a stochastic [1]) matrix $Q=(q_{ij})$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)} = q_{ij} \qquad (i, j=1, 2, \ldots, N).$$

Moreover $QP=PQ=Q$ and $Q^2=Q$.

However, by a counter example we shall see that this theorem cannot be proven from the intuitionistic point of view.

3.3.2. Counter-example.

Let ϱ be a real number for which we have no proof of

$$(\varrho=0) \vee (\varrho \# 0)$$

and let P be the matrix:

$$P = \begin{pmatrix} 1-\varrho & \varrho \\ 0 & 1 \end{pmatrix}.$$

It is easily seen that

$$p_{11}^{(n)}=(1-\varrho)^n \quad ; \quad p_{12}^{(n)}=1-(1-\varrho)^n \quad ; \quad p_{21}^{(n)}=0 \quad ; \quad p_{22}^{(n)}=1.$$

From this result it becomes clear that if we have a proof of $\varrho \# 0$, then

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

but if we have a proof of $\varrho=0$, then

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies that if $\varrho \# 0$ the limit occurring in theorem 3.3.1. becomes:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{11}^{(m)} = \lim \frac{1}{n} \sum_{m=1}^{n} (1-\varrho)^m = 0,$$

but if $\varrho=0$ then this limit becomes:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{11}^{(n)} = \lim \frac{1}{n}\, n = 1.$$

---

[1]) A matrix $Q=(q_{ij})$ is called a stochastic matrix if $\sum_{j=1}^{N} q_{ij}=1$ and $q_{ij} \nless 0$ for all i and j.

This illustrates the fact that there cannot exist (nowadays) a proof of theorem 3.3.1. The example shows a Markov Chain for which the sequence $\{1/n \sum_1^n p_{11}^{(m)}\}$ is twofold negatively convergent (for this notion of convergence cf. [10]).

3.3.3. The construction of the foregoing counterexample is based on the possibility that we can construct real numbers ϱ for which we have no proof of the disjunction

$$(\varrho = 0) \vee (\varrho \# 0).$$

This construction of a counter example just exhausts all possibilities in the sense that if $P = (p_{ij})$ is a finite stochastic matrix such that

$$(1) \qquad (i, j)\ [(p_{ij} = 0) \vee (p_{ij} \# 0)]$$

then theorem 3.3.1. can be proven from the intuitionistic point of view.

Note that condition (1) implies

$$(i, j)\ [(p_{ij} = 1) \vee (p_{ij} \# 1)].$$

Doob (cf. [6], p. 175) gives two proofs of theorem 3.3.1. The first proof where an analytical method is used does not hold from our point of view but the second proof (cf. [6], p. 176–181), where the possible cases are discussed, can be rewritten with only slight modifications, which are evident. The method of Doob's second proof gives much more information about the behaviour of a Markov Chain with a finite number of states and under condition (1) it becomes clear that every state is either transient or non-transient (cf. [6], p. 178) and the species of all states can be divided into the class of transient states and the ergodic classes (cf. [6], p. 178), which classes are disjoint.

3.3.4. The matrix defined in section 3.3.2. does not satisfy condition (1) (3.3.3.) and theorem 3.3.1. could not be proven. Now we prove that it is already sufficient if we know that the sequence $\{1/n \sum_{m=1}^n p_{ij}^{(m)}\}$ are negatively convergent. (For this notion of convergence cf. [4] or [10]).

Theorem. Let $P = (p_{ij})$ be a stochastic matrix and let us suppose that the limits

$$\neg\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)} \overset{\text{df}}{=} q_{ij}$$

exist for every i and j then the matrix $Q = (q_{ij})$ is a stochastic matrix and $PQ = QP = Q = Q^2$.

Proof. We define the real numbers ${}_n\sigma_{ij}$ by

$${}_n\sigma_{ij} = \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)}.$$

From the implication

$$\left(-\frac{\varepsilon}{N} < {}_n\sigma_{ij} - q_{ij} < \frac{\varepsilon}{N}\right) \Rightarrow \left(-\varepsilon < 1 - \sum_{j=1}^{N} q_{ij} < \varepsilon\right)$$

the following implications are easily seen:

$$(\varepsilon) \neg\neg (\mathcal{I}k_{ij})\left(n > k_{ij} \Rightarrow |{}_n\sigma_{ij} - q_{ij}| < \frac{\varepsilon}{N}\right) \Rightarrow$$

$$(\varepsilon) \neg\neg (\mathcal{I}k_{ij})\left(\left|1 - \sum_{j=1}^{N} q_{ij}\right| < \varepsilon\right) \Rightarrow (\varepsilon)\left(\left|1 - \sum_{j=1}^{N} q_{ij}\right| < \varepsilon\right) \Rightarrow \sum_{j=1}^{N} q_{ij} = 1.$$

Using matrix notation we have furthermore:

$$\overline{\lim_{n\to\infty}}\, \frac{1}{n} \sum_{m=1}^{n} P^{(m)} = Q \Rightarrow \overline{\lim_{n\to\infty}}\, \frac{1}{n} \sum_{m=1}^{n} P^{(m+1)} = PQ = QP \Rightarrow$$

$$\overline{\lim_{n\to\infty}} \left[\frac{1}{n} \sum_{m=1}^{n+1} P^{(m)} - \frac{1}{n} P\right] = PQ \Rightarrow$$

$$\overline{\lim}\, \frac{1}{n} \sum_{m=1}^{n+1} P^{(m)} = PQ \Rightarrow \overline{\lim}\, \frac{n+1}{n}\, \overline{\lim}\, \frac{1}{n+1} \sum_{m=1}^{n+1} P^{(m)} = PQ \Rightarrow Q = PQ.$$

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